Locus Problems When Reversing Traveling Direction

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Abstract

It is often a challenging task if we are asked to find the locus that depends on other curves, but it can become intuitive if we use a technological tool to find the trace of the locus first and later find the equation analytically. In this paper, we shall see that when we reverse the direction of traveling for a point on one curve, and ask the locus for the corresponding curve, the task becomes interesting and non-trivial. In many instances, since we have involved several parameters when constructing a locus curve, it is often that we may construct a family of new parametric curves from the existing closed curve. We will also outline ways of extending locus curves found in 2D to corresponding ones in 3D.

1 Introduction

Finding a curve defined by the locus of a moving point has been popular and is often asked on Gaokao (a college entrance exam) in China. There have been several exploratory activities (see [6] [9], and [11]) derived from Chinese college entrance exam practice problems ([10]). In Section 2, we are asked to find the locus that is determined by moving points on some respective circles, when suitable mild conditions are imposed; in particular, one of the moving points is traveling in the opposite direction (clockwise direction). In Section 3, we look for the locus that is produced by linear combinations of vectors, where points on respective curves might be traveling in reverse directions. Surprisingly, we are able to create a new family of interesting graphs by using such construction. In Section 4, we see how we can generate interesting locus curves when points are traveling in certain directions along three respective curves. As a result, cardioid-shape curves can be generated. (See Figures 6(a) or 6(b).) In Section 5, we attempt to generalize our locus curves from 2D into corresponding 3D locus surfaces (we shall do this either by a suitable rotation of the 2D locus curve or by considering some ellipsoids as "generated" by tangent spheres rotating around big circles).

Results in this paper rely on extensive and nontrivial use of parametric equations of curves and surfaces in 2D and 3D but, with the help of technological tools, it is natural to explore the trace of a possible locus before attempting to find a complete analytic or algebraic solution is much more accessible, convincing and intuitive to students. In this paper, in addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of a dynamic geometry system (DGS) such as ClassPad Manager [2] or GeoGebra [3]. Secondly, we ask for a symbolic answer if possible, by using a symbolic geometry system (SGS) such as Geometry Expressions [4]. Finally, we use a computer algebra system (CAS) such as Maple [7] or maxima [8] to verify that our analytic solutions are identical to those obtained by using SGS.

2 Locus Of Linear Combinations

Learning parametric equations is quite important in many high school or college curriculum. We have seen the importance of using vectors when finding the locus of moving points in [1]. **The locus curve becomes non-trivial if we reverse the direction of traveling for the moving point on a curve when finding the locus**.

In order to gain familiarity and intuition about the effect reversing the direction of travel around a reference curve has on the corresponding locus, let us consider the locus generated by the midpoint of rotating point A on unit circle C_1, and rotating points B and B' on circle C_2 centered at the origin and with given radius $r > 1$. It's clear that when A and B travel both in counterclockwise direction, locus generated by midpoint M is the circle C centered at the origin and with radius $(r+1)/2$, but for A traveling in counterclockwise direction and B' traveling in clockwise direction, locus generated by midpoint M' is the ellipse centered at the origin and with axis of length $r+1$ and $r-1$. Exploration can be found in [S1].

As a second example, suppose we are given three circles C_1, C_2 and C_3 that are shown in Figure 1(a). We assume C_1, C_2 and C_3 centered at $A = (0,0), B = (a,b), C = (c,d)$ and radii of r_1, r_2 and r_3 respectively. Let D, E and F be three moving points on these three circles, C_1, C_2 and C_3, traveling all in the counterclockwise direction, respectively. We are interested in finding the locus G satisfying

$$\overrightarrow{DG} = r\overrightarrow{DE} + s\overrightarrow{DF}.$$

Then it is not difficult to prove the locus of G is a circle, where r and $s \in (0,1)$. We note that $\overrightarrow{OG} = \overrightarrow{OD} + r\overrightarrow{DE} + s\overrightarrow{DF}$. We write $\begin{bmatrix} x_1(r_1,t) \\ y_1(r_1,t) \end{bmatrix} = \begin{bmatrix} r_1 \cos t \\ r_1 \sin t \end{bmatrix}$, $\begin{bmatrix} x_2(a,r_2,t) \\ y_2(b,r_2,t) \end{bmatrix} = \begin{bmatrix} a + r_2 \cos t \\ b + r_2 \sin t \end{bmatrix}$ and $\begin{bmatrix} x_3(c,r_3,t) \\ y_3(d,r_3,t) \end{bmatrix} = \begin{bmatrix} c + r_3 \cos t \\ d + r_3 \sin t \end{bmatrix}$. Then $\overrightarrow{OG} = \overrightarrow{OD} + r\overrightarrow{DE} + s\overrightarrow{DF}$ can be written as

$$\begin{bmatrix} x_4(a,c,r_1,r_2,r_3,r,s,t) \\ y_4(b,d,r_1,r_2,r_3,r,s,t) \end{bmatrix} = \begin{bmatrix} x_1(r_1,t) \\ y_1(r_1,t) \end{bmatrix} + r \begin{bmatrix} x_2(a,r_2,t) - x_1(r_1,t) \\ y_2(b,r_2,t) - y_1(r_1,t) \end{bmatrix} + s \begin{bmatrix} x_3(c,r_3,t) - x_1(r_1,t \\ y_3(d,r_2,t) - y_1(r_1,t) \end{bmatrix}.$$

After simplifying, we see

$$\begin{bmatrix} x_4(a,c,r_1,r_2,r_3,r,s,t) \\ y_4(b,d,r_1,r_2,r_3,r,s,t) \end{bmatrix} = ((-r-s+1)\,r_1 + rr_2 + sr_3) \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \begin{bmatrix} ar + cs \\ br + ds \end{bmatrix},$$

which represents a circle for $t \in [0, 2\pi], r$ and $s \in (0,1)$. We depict three circles when $a = 1, b = -1, c = 2, d = 0.5, r_1 = 2, r_2 = r_3 = 1$ in Figure 1(a), and the locus in red when $r = \frac{1}{2}$ and $s = 1$ in

Figure 1(b))

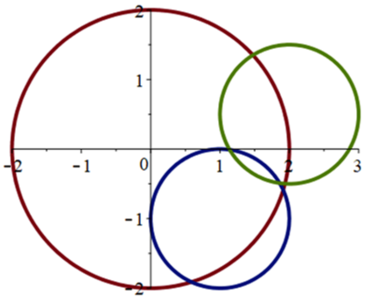

Figure 1(a) Three moving points on three respective cirlces.

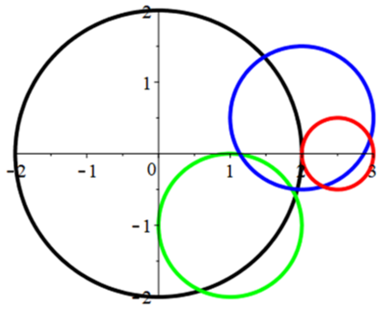

Figure 1(b). Locus in red when $r = \frac{1}{2}$ and $s = 1$

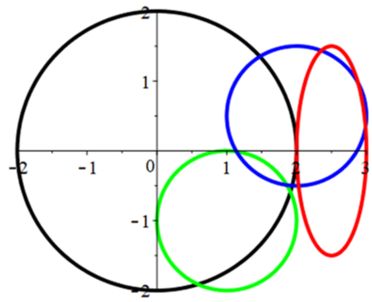

Figure 1(c). Locus in red when F is traveling clockwise

Main Problem: *It is interesting to see that the locus G, satisfying $\overrightarrow{DG} = r\overrightarrow{DE} + s\overrightarrow{DF}$, where r and $s \in [0,1]$, will not be a circle when we choose the point D, E or F to be traveling in a clockwise direction on three circles, C_1, C_2 and C_3* respectively.

For example we consider the scenario when moving points D and E travel in the counterclockwise direction but F travels in the clockwise direction. We shall show that the locus to be an ellipse. We write $D = (r_1 \cos t, r_1 \sin t)$, $E = (a + r_2 \cos t, b + r_2 \sin t)$ and $F = (c + r_3 \cos t, d - r_3 \sin t)$. We see the locus $\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{DG}$, where

$$\begin{aligned}\overrightarrow{DG} = \overrightarrow{DE} + \overrightarrow{DF} &= (a + (r_2 - r_1)\cos t, b + (r_2 - r_1)\sin t) \\ &+ (c + (r_3 - r_1)\cos t, d - (r_3 + r_1)\sin t) \\ &= ((a + c) + (r_2 + r_3 - 2r_1)\cos t, (b + d) + (r_2 - r_3 - 2r_1)\sin t).\end{aligned}$$

This implies that the locus $\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{DG}$

$$\begin{aligned}&= (r_1 \cos t, r_1 \sin t) + \\ &((a + c) + (r_2 + r_3 - 2r_1)\cos t, (b + d) + (r_2 - r_3 - 2r_1)\sin t) \\ &= ((a + c) + ((r_2 + r_3 - r_1)\cos t), (b + d) + (r_2 - r_3 - r_1)\sin t)\end{aligned}$$

to be an ellipse shown as follows:

$$\left(\frac{x - (a + c)}{r_2 + r_3 - r_1}\right)^2 + \left(\frac{y - (b + d)}{r_2 - r_3 - r_1}\right)^2 = 1.$$

We depict three circles when $a = 1, b = -1, c = 2, d = 0.5, r_1 = 2, r_2 = r_3 = 1$ in Figure 1(a), and the locus in red when $r = \frac{1}{2}$ and $s = 1$ in Figure 1(c). More exploration can be found in [S2]. We present the following scenario of finding the locus when reversing the traveling of one point.

2.1 Locus of tangent circles

We consider a fixed circle C_1 that is centered at A with radius r_1 (see the circle in black in the Figure 2(a) or 2(b)) and the second circle C_2 with fixed radius r_2 and $r_2 < r_1$ that is centered at B (see the circle in blue in the Figure 2(a) or 2(b)) and is tangent to C_1 at the point of tangency E. We let the point E travel in the counterclockwise direction. Let F be a moving point on C_2 and move in the clockwise direction, we shall find the locus for the point F. We note that the Figure 2(a) shows the circle C_2 is interiorly tangent to the circle C_1 while the Figure 2(b) shows the circle C_2 is exteriorly to the circle C_1. We shall show that the locus F in both cases are ellipses.

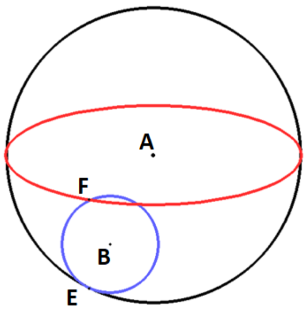

Figure 2(a). One circle is moving insribed within the other circle.

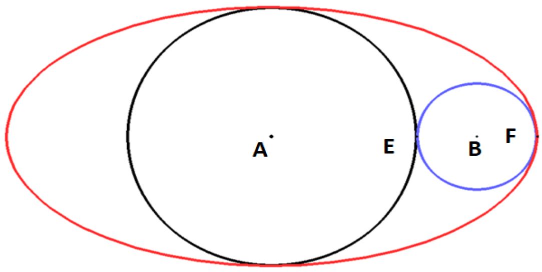

Figure 2(b). One circle is moving outside the other circle.

Without loss of generality, we assume the circle at A is $x^2 + y^2 = r_1^2$ and the circle centered at B is $(x-a)^2 + (y-b)^2 = r_2^2$, where $r_1 > r_2$. We write $F = (x, y)$, and we want to find $\overrightarrow{OF} = \overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF}$.

Example 1 *Suppose the circle C_2 is interiorly tangent to the circle C_1. Then, since $E \in C_1 \cap C_2$, we write $E = (r_1 \cos t, r_1 \sin t) = (a+r_2 \cos t, b+r_2 \sin t)$ and note that $\overrightarrow{EF} = (x-r_1 \cos t, y-r_1 \sin t)$. Therefore, $\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = \begin{bmatrix} a + r_2 \cos t \\ b + r_2 \sin t \end{bmatrix} + \begin{bmatrix} x - r_1 \cos t \\ y - r_1 \sin t \end{bmatrix} = \begin{bmatrix} a + x + (r_2 - r_1) \cos t \\ b + y + (r_2 - r_1) \sin t \end{bmatrix}$. Since F is traveling clockwise at the circle C_2, it satisfies the equation $\begin{bmatrix} a + r_2 \cos(-t) \\ b + r_2 \sin(-t) \end{bmatrix} = \begin{bmatrix} a + r_2 \cos t \\ b - r_2 \sin t \end{bmatrix}$, which implies*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_2 \cos t - (r_2 - r_1) \cos t \\ -r_2 \sin t - (r_2 - r_1) \sin t \end{bmatrix} = \begin{bmatrix} r_1 \cos t \\ (-2r_2 + r_1) \sin t \end{bmatrix}.$$

Therefore, the locus satisfies the equation of the ellipse as shown below

$$\left(\frac{x}{r_1}\right)^2 + \left(\frac{y}{r_1 - 2r_2}\right)^2 = 1.$$

Example 2 *Suppose the circle C_2 is exteriorly tangent to the circle C_1. Then, when $t = 0$, E is at $t = \pi$ of C_2, therefore, we need to write $E = (r_1 \cos t, r_1 \sin t) = (a + r_2 \cos(t + \pi), b +$*

$r_2 \sin(t+\pi)) = (a - r_2\cos t, b - r_2 \sin t)$ *and note that* $\overrightarrow{EF} = (x - r_1\cos t, y - r_1 \sin t)$. *Therefore,* $\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = \begin{bmatrix} a + r_2\cos(t+\pi) \\ b + r_2 \sin(t+\pi) \end{bmatrix} + \begin{bmatrix} x - r_1\cos t \\ y - r_1 \sin t \end{bmatrix} = \begin{bmatrix} a + x - r_1\cos t - r_2 \cos t \\ b + y - r_1 \sin t - r_2 \sin t \end{bmatrix}$. *Since* F *is traveling clockwise at the circle* C_2, *it satisfies the equation of* $\begin{bmatrix} a + r_2\cos(-t) \\ b + r_2\sin(-t) \end{bmatrix} = \begin{bmatrix} a + r_2\cos t \\ b - r_2 \sin t \end{bmatrix}$, *which implies*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_2\cos t \\ -r_2\sin t \end{bmatrix} - \begin{bmatrix} -r_1\cos t - r_2\cos t \\ -r_1 \sin t - r_2 \sin t \end{bmatrix} = \begin{bmatrix} (2r_2 + r_1)\cos t \\ r_1 \sin t \end{bmatrix}$$

Therefore, the locus satisfies the equation of the ellipse as shown below

$$\left(\frac{x}{2r_2 + r_1}\right)^2 + \left(\frac{y}{r_1}\right)^2 = 1.$$

The Figures 2(a) and 2(b) show when $r_1 = 3$ and $r_2 = 1$ in both cases. We leave the following similar problems as an exercise for readers to explore.

Exercise 3 *We first fix the circle* C_1 *(see the blue circle in Figure 3(a) or 3(b)). Next, the second moving circle* C_2 *is such that* C_1 *is inscribed in the circle* C_2 *and tangent at a point* B. *Lastly, the circle* C_3 *is in the exterior of the circle* C_1 *and tangent at* B *also. Let both circles* C_2 *and* C_3 *be moving in the counterclockwise direction. If the point* C *is moving in the clockwise direction on* C_3, *we need to find the locus* $\overrightarrow{OC}$ *in this case.*

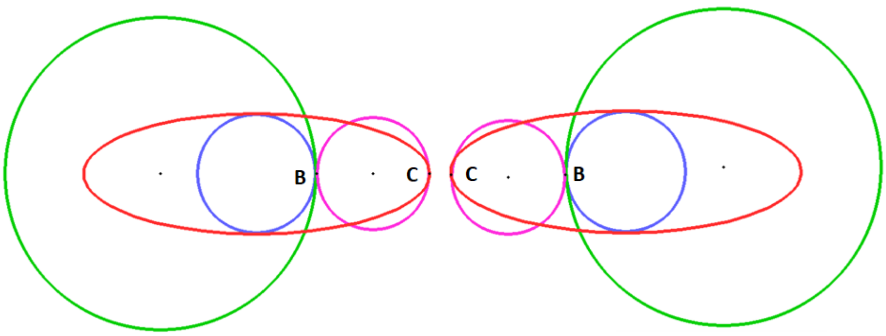

Figure 3(a). Locus G *when* $t = 0$ *Figure 3(b). Locus* G *when* $t = \pi$

The Figures 3(a) and 3(b) above show the locus in red when the equation for the circle C_1 is $x^2 + y^2 = 1$, the radii for the circles C_2 and C_3 are about 2.610153 and 0.9447343 respectively.

3 Generating Families Of Interesting Curves

We first present an example that shows the locus obtained from linear combinations of vectors can create families of interesting closed curves. Next, we discuss if we reverse the direction of traveling for one curve, we shall see some unexpected interesting locus curves.

Example 4 *We consider the ellipse of* $C_1, [x_1(t), y_1(t)] = [2\cos t, \sin t)]$, *the cardioid of* $C_2, [x_2(t), y_2(t)] = [(1-\cos t)\cos t, (1-\cos t)\sin t]$ *and the circle of* $C_3, [x_3(t), y_3(t)] = [\cos t, \sin t]$, *where* $t \in [0, 2\pi]$. *If* I, F *and* G *are moving points on* $[x_1(t), y_1(t)], [x_2(t), y_2(t)]$ *and* $[x_3(t), y_3(t)]$ *respectively, traveling all in counterclockwise directions. What is the locus* J, *that is on* $r\overrightarrow{IF} + s\overrightarrow{IG}$, *where* r, s *are real numbers?*

It is trivial to see the locus J satisfies

$$\overrightarrow{OJ} = \overrightarrow{OI} + r\overrightarrow{IF} + s\overrightarrow{IG},$$

where. Thus if $[x_4(t), y_4(t)]$ represents the locus curve of C_4, we have $\begin{bmatrix} x_4(r,s,t) \\ y_4(r,s,t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} + r\begin{bmatrix} x_2(t) - x_1(t) \\ y_2(t) - y_1(t) \end{bmatrix} + s\begin{bmatrix} x_3(t) - x_1(t) \\ y_3(t) - y_1(t) \end{bmatrix}$. We depict the graph for $\begin{bmatrix} x_4(1,1,t) \\ y_4(1,1,t) \end{bmatrix}$, $t \in [0, 2\pi]$, which represents a double folium in blue in Figure 4(a). We see an interesting three-leaf rose if we use $\begin{bmatrix} x_4(2,1,t) \\ y_4(2,1,t) \end{bmatrix}$,where $t \in [0, 2\pi]$, in blue in Figure 4(b).

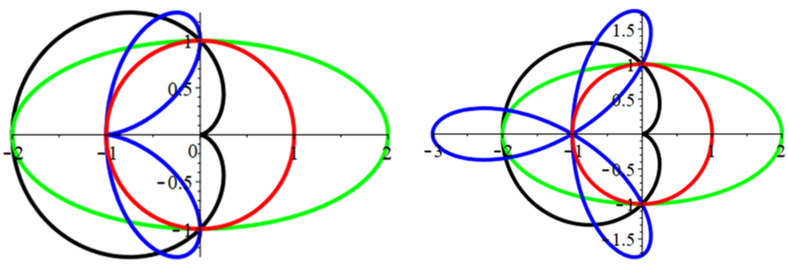

Figure 4(a). A double folium in blue. Figure 4(b). A three-leave rose in blue

Now we ask for the locus if we reverse the traveling direction for any one of the parametric curves C_1, C_2 or C_3. For example, suppose we reverse the traveling direction for $[x_1(t), y_1(t)]$ from counterclockwise to clockwise, then its parametric equation is written as $[x_1(-t), y_1(-t)]$, where $t \in [0, 2\pi]$. As a result, the locus is

$$\begin{bmatrix} x_4^*(r,s,t) \\ y_4^*(r,s,t) \end{bmatrix} = \begin{bmatrix} x_1(-t) \\ y_1(-t) \end{bmatrix} + r\begin{bmatrix} x_2(t) - x_1(-t) \\ y_2(t) - y_1(-t) \end{bmatrix} + s\begin{bmatrix} x_3(t) - x_1(-t) \\ y_3(t) - y_1(-t) \end{bmatrix},$$

where $t \in [0, 2\pi]$, r and s are real numbers. Explorations can be found in [S3]. We pose the following interesting questions, whose Maple explorations can be found in [S3] also.

3.1 Open questions

1. For real numbers r and s, consider the family of curves for $[x_4^*(r,s,t), y_4^*(r,s,t)]$ in the preceding Example, find the best values r and s so that (a) the plot for $[x_4^*(r,s,t), y_4^*(r,s,t)]$ encloses

all the plots from $[x_1(t), y_1(t)]$, $[x_2(t), y_2(t)]$ and $[x_3(t), y_3(t)]$. (b) The plot of $[x_4^*(r,s,t), y_4^*(r,s,t)]$ leaves the smallest gap for $C_4-(C_1 \cup C_2 \cup C_3)$. We describe two possible scenarios, $\left[x_4^*(\frac{1}{4}, -.55072, t), y_4^*(\right.$ and $\left[x_4^*(\frac{1}{8}, -.37681, t), y_4^*(\frac{1}{8}, -.37681, t)\right]$ that are shown in green color of the Figures 5(a) and 5(b) respectively.

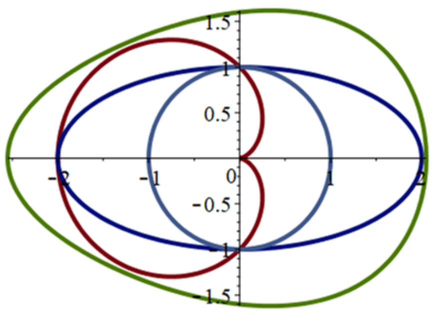

Figure 5(a). Plot for $s = -0.55072$ of $\left[x_4(\frac{1}{4}, -s, t), y_4(\frac{1}{4}, -s, t)\right]$ in green

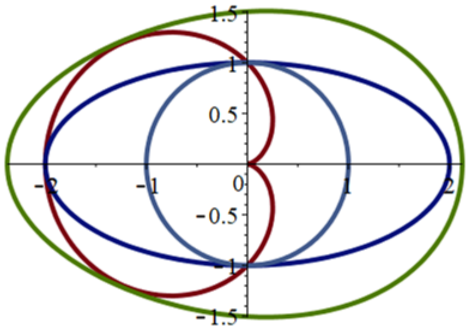

Figure 5(b). Plot for $s = -.37681$ of $\left[x_4(\frac{1}{8}, s, t), y_4(\frac{1}{8}, s, t)\right]$ in green

2. Here we consider the following question: We are given three closed curves C_1, C_2 and C_3 represented by $[x_1(t), y_1(t)]$, $[x_2(t), y_2(t)]$ and $[x_3(t), y_3(t)]$, respectively. If I, F and G are moving points, which can be in counterclockwise or clockwise direction on any one of the C_1, C_2 and C_3 respectively. Let $[x_4(r,s,t), y_4(r,s,t)]$ represent the locus J such that $\overrightarrow{OJ} = \overrightarrow{OI} + r\overrightarrow{IF} + s\overrightarrow{IG}$, where r, s are real numbers. Does there always exist r and s such that the plot of $[x_4(r,s,t), y_4(r,s,t)]$ which will leave the smallest gap for $C_4 - (C_1 \cup C_2 \cup C_3)$?

3. Extend our observations in 2D, mentioned in item 2 above, to corresponding ones in 3D.

4 Generating Interesting Curves With Three Circles

In view of Section 2.1, in which the locus generated is an ellipse when two circles are considered, we explore the locus when three circles are considered. Explicitly, in this section we find the parametric equation of the curve corresponding to the locus generated by a point that moves in a "circle" C_3, arranged in a similar way to a spirograph together with circles C_1 and C_2, as described below.

4.1 Circle C_3 exteriorly tangent to circle C_2

In the simplest case, we can assume that C_1, shown in blue in the figures of this section, is the unit circle $x^2 + y^2 = 1$. The circle C_2 of radius r_2 and moving center A, shown in black in the figures of this section, rotates in counterclockwise direction keeping externally tangent to C_1 at the point $B = (\cos(\theta), \sin(\theta))$, for $\theta \in [0, 2\pi]$. Next, for r and γ given let $E = r(\cos(\gamma), \sin(\gamma))$ be a point "exterior" to C_2, and let F_1 and F_2 be the points of intersection of the circle C_2 with the line AE.

The first problem is to describe the locus of a point G_+ traveling in counterclockwise direction along the "circle" C_3, shown in orange in figures of this section. In fact,

$$G_+ = E + d_{r_2,r,\gamma}(\theta)(\cos(\theta), \sin(\theta))$$

travels along a "family of circles" C_3 of variable radius, such that each one of them is centered at E and is tangent *exteriorly* to C_2 at $F \doteq F_1$, the point between A and E. To keep the exteriorly tangency condition, point E must be chosen outside the circle C, shown in light grey in Figure 6(a) and 6(b), which is centered at the origin and has radius $2r_2 - 1$.

For example, the curve $g_{r_2,r,\gamma,+}$ corresponds to the locus generated by G_+ when $r_2 = 3$ and $r = 5$, and is shown in red in Figures 6(a) and 6(b) for $\gamma = \pi/4$ and $\gamma = 3\pi/4$, respectively.

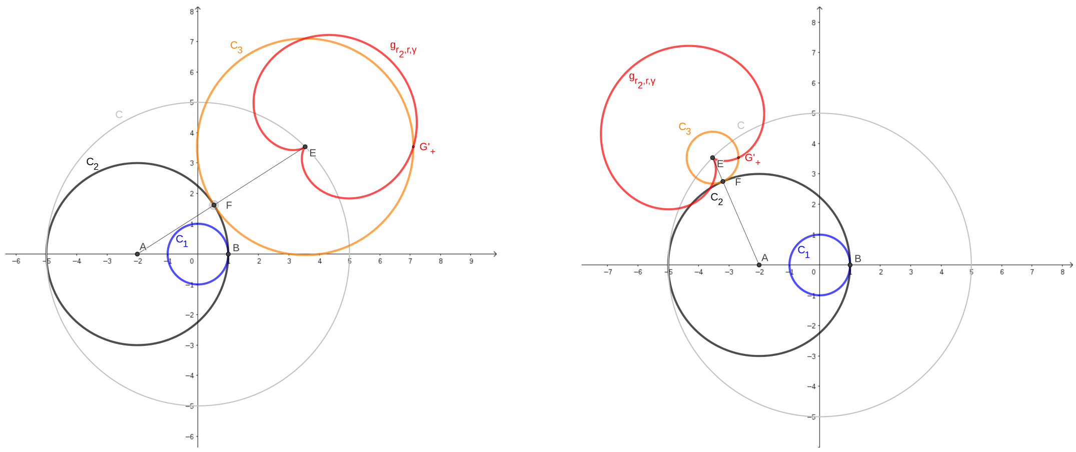

Figure 6(a). Locus generated by G_+ for $E(r = 5; \gamma = \pi/4)$.

Figure 6(b). Locus generated by G_+ for $E(r = 5; \gamma = 3\pi/4)$.

Let $g_{r_2,r}$ denote the curve for the locus generated by G_+ when $\gamma = 0$ (shown in red and dotted style in the figures of this section). It is clear that $g_{r_2,r,\gamma}$ can be obtained by a rotation of this curve around the origin by the angle γ, as shown in Figure 7.

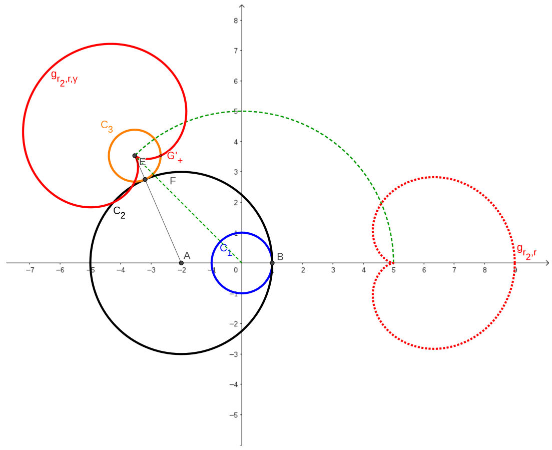

Figure 7. Locus generated by G_+ as a rotation of the curve $g_{r_2,r}$ corresponding to $E = (r = 5; \gamma = 0)$.

The previous result motivates us to study a more general setting: first we apply a scale factor e, so that now C_1 and C_2 have, respectively, radius e and er_2, and C_2 rotates in counterclockwise direction keeping externally tangent to C_1 at the point $B = e(\cos(\alpha+\theta), \sin(\alpha+\theta))$, for $\theta \in [0, 2\pi]$. Note that initially, when $\theta = 0$, the line segment OB makes an angle α with the x–axis. We consider now the point $E = er(\cos(\gamma), \sin(\gamma))$ that must be chosen outside the circle C centered at the origin and with radius $e(2r_2 - 1)$. Our problem becomes in finding the locus generated by the point

$$G'_+ = E + d_{r_2,r,\gamma,e,\alpha,\beta}(\theta)(\cos(\beta+\theta), \sin(\beta+\theta))$$

Note that initially, when $\theta = 0$, the line segment EG'_+ makes an angle β with the x–axis.

In Figure 8(a) we see that a translation of $g_{r_2,r,\gamma}$ by the vector $(e-1)r(\cos(\gamma), \sin(\gamma))$, that we called $g_{r_2,r,\gamma,e}$, followed by a dilation (or contraction) with respect to the point E by a factor e, produces the curve denoted by $g'_{r_2,r,\gamma,e}$. Finally, as shown in Figure 8(b), a rotation of $g'_{r_2,r,\gamma,e}$ around the point E by the angle $\beta - \alpha$ produces the curve $g'_{r_2,r,\gamma,e,\alpha,\beta,+}$ corresponding to the locus generated by G'_+. Explorations can be found in [S4].

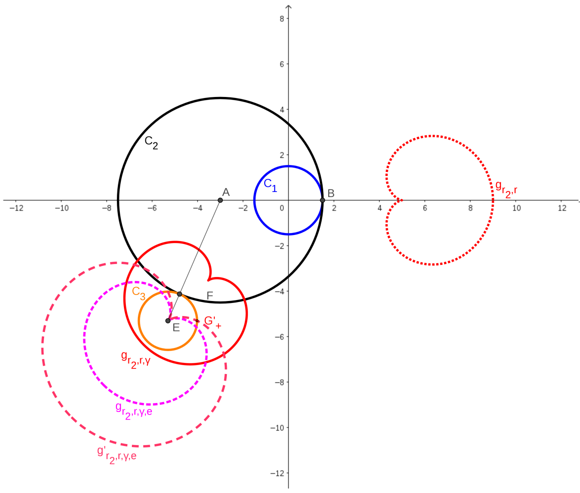

Figure 8(a). Dilation by $e = 3/2$ when $E = (r = 5; \gamma = 5\pi/4)$.

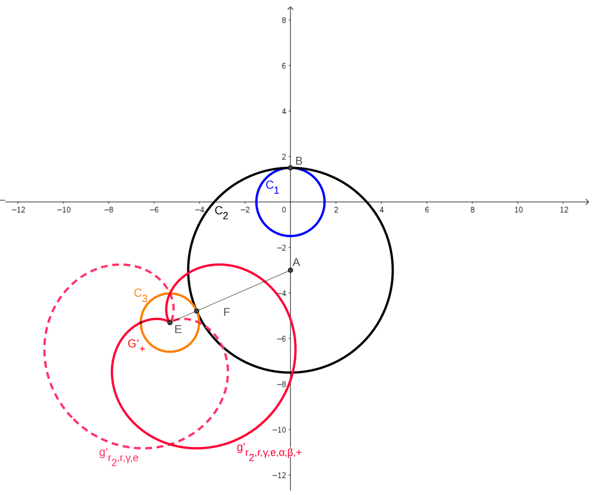

Figure 8(b).Rotation when $\alpha = \pi/2$ and $\beta = \pi$.

For what said in the previous paragraphs, it is enough to determine the parametric equation of the curve $g_{r_2,r}$. With the help of the CAS maxima (exploration can be found in [S5]), we get

$$g_{r_2,r}(t) = (r, 0) + d_{r_2,r}(t)(\cos(t), \sin(t)), \quad t \in [0, 2\pi]$$

where

$$d_{r_2,r}(t) = \sqrt{-2r_2\sqrt{2r(r_2-1)\cos(t) + r^2 + (r_2-1)^2} + r_2^2 + 2r(r_2-1)\cos(t) + r^2 + (r_2-1)^2}$$

A natural question is: what happens by considering a point G_- traveling now in clockwise direction along the "circle" C_3? For example, the curve $g'_{r_2,r,\gamma,e,\alpha,\beta,-}$, shown in green in Figure 9, corresponds to the locus generated by G_- for the values of the parameters used previously. We see that a rotation of $g'_{r_2,r,\gamma,e}$ around the point E by the angle $\beta + \alpha - 2\gamma$ produces the curve $g'_{r_2,r,\gamma,e,\alpha,\beta,-}$ (see [S4]).

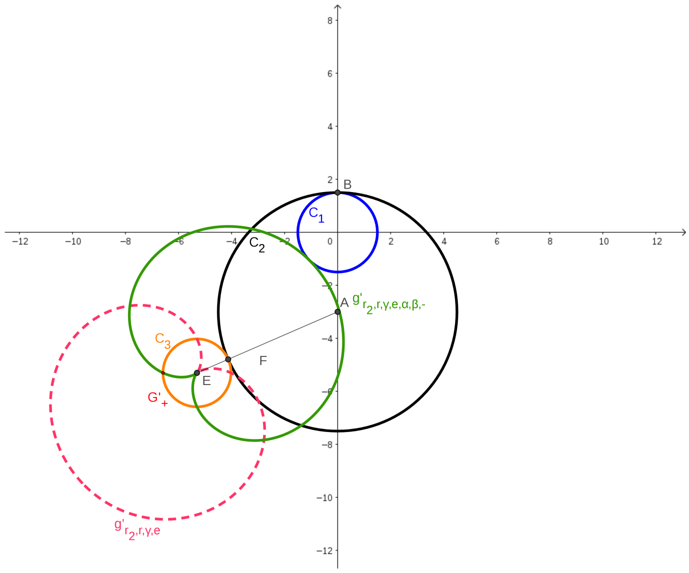

Figure 9. Locus generated by G traveling in clockwise direction along C_3.

4.2 Circle C_2 interiorly tangent to circle C_3

The second problem is to describe the locus of points H_+ and H_- traveling, respectively, in counter-clockwise and clockwise direction along the "circle" C_3', shown in orange in Figure 10. In fact,

$$H_\pm = E + d'_{r_2,r,\gamma,e,\alpha,\beta}(\theta)(\cos(\beta \pm \theta), \sin(\beta \pm \theta))$$

travels along a "family of circles" C_3' of variable radius, such that each one of them is centered at E with C_2 tangent *interiorly* to C_3' at $F \doteq F_2$, the antipodal point to $F1$ on C_2.

Let $h_{r_2,r}$ denotes the curve for the locus generated by H_+ when $e = 1$ and $\alpha = \beta = \gamma = 0$. When the sequence of plane transformations used in our first problem are applied to this curve, we get the curve $h'_{r_2,r,\gamma,e,\alpha,\beta,+}$ corresponding to the locus generated by H_+. Explorations can be found in [S6]. So, it is enough to determine the parametric equation of the curve $h_{r_2,r}$ ((exploration can be found in [S7])), that turns out to be

$$h_{r_2,r}(t) = (r, 0) + d'_{r_2,r}(t)(\cos(t), \sin(t)), \quad t \in [0, 2\pi]$$

where

$$d'_{r_2,r}(t) = \sqrt{2r_2\sqrt{2r(r_2-1)cos(t) + r^2 + (r_2-1)^2} + r_2^2 + 2r(r_2-1)cos(t) + r^2 + (r_2-1)^2}$$

Finally, keeping fixed the values of the parameters, let us consider the curve $h'_{r_2,r,\gamma,e,\alpha,\beta,-}$ that corresponds to the locus generated by H_-. Again, a rotation of $h'_{r_2,r,\gamma,e}$ around the point E by the angle $\beta + \alpha - 2\gamma$ produces the curve $h'_{r_2,r,\gamma,e,\alpha,\beta,-}$ (see [S6]).

For example, the curves $h'_{r_2,r,\gamma,e,\alpha,\beta,\pm}$, shown in red/green in Figure 10, correspond to the locus generated by $H_{\pm}$ when $r_2 = 3,\ r = 5,\ \gamma = 7\pi/4,\ e = 3/2,\ \alpha = \pi/2$ and $\beta = \pi$.

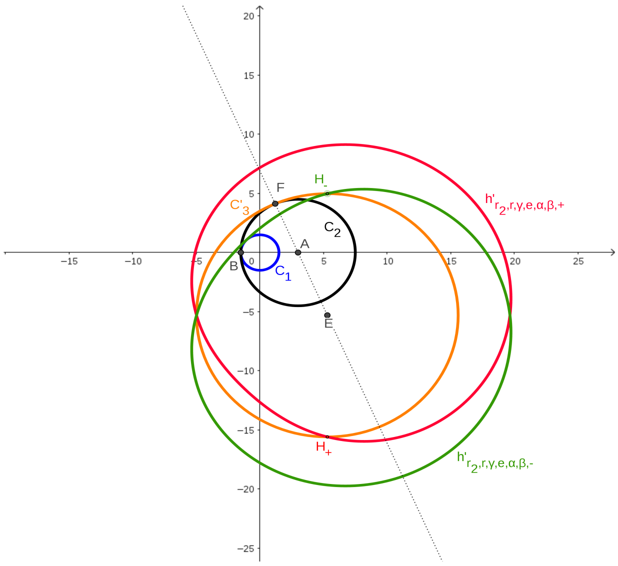

Figure 10. Locus generated by H_+ and $H-$.

As a final remark, in future work we will study the inverse problems: that is, if we are given the locus curves for G and H, respectively, find a way to construct three circles with all appropriate needed parameters in order to reproduce the locus curves given.

5 3D Locus Surface By A Rotation Or Projection

We would like to extend our 2D locus curves discussed in the preceding sections to corresponding 3D locus surfaces. We shall do this either by a rotation or a projection.

5.1 Surfaces generated by rotation

Since locuses generated in the preceding sections are (plane transformations of) curves that are symmetric with respect to the x–axis, it's easy to visualize (but may be not so simple to graph) the surface of revolution obtained by a rotation around x–axis of a given 2D locus curve (ellipsoids for the cases considered in Section 2 and "heart shaped" surfaces for the cases in Section 4). Here we show how a DGS with 3D graphics capabilities, in this case GeoGebra, can be used to construct these surfaces of revolution, even without knowing the equation for the base 2D curve. To make these constructions we take advantage of the dynamic features of this kind of software, creating animations in 3D of the locus generated by a point moving in planes that can be rotated around the x–axis by an angle $\phi \in [0, \pi]$ (in red in the figures of this section); the resulting trajectories produce a frame–wired style 3D graph that simulates the surface of revolution sought. Exploration can be found in [S8]. As an example, Figures 11(a) and 11(b) correspond to the case when circle C_2 with radius $r_2 = 3$ turns exteriorly around the circle C_1 with radius $r_1 = 4$ and centered at the origin.

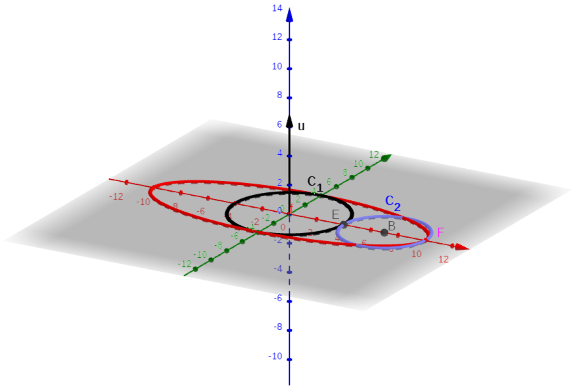

Figure 11(a). Ellipse on the plane $z = 0$ ($\phi = 0$).

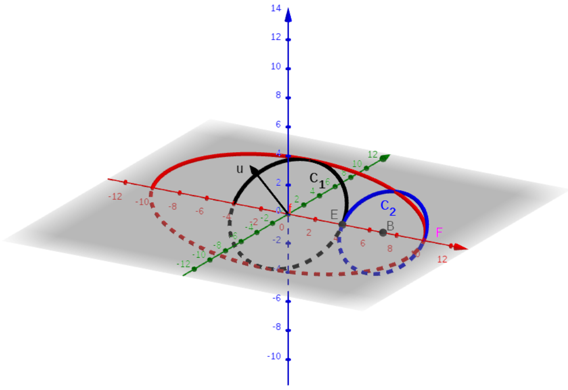

Figure 11(b). Ellipse on the plane $z = y$ ($\phi = \pi/4$).

Then, we use a CAS, in this case maxima, to determine the parametric equation of the surface of revolution under study, that in this case turns to be (exploration can be found in [S9]).

$$S_{r_1,r_2} = ((r_1 + 2r_2)\cos(u), r_1\cos(v)\sin(u), r_1\sin(v)\sin(u)); \quad u \in [0, 2\pi],\ v \in [0, \pi]$$

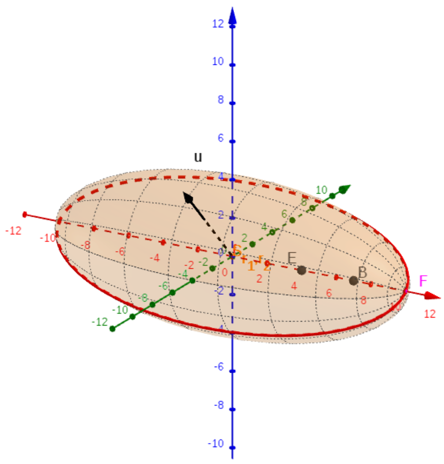

Figure 12. Ellipsoid of revolution.

If we know the parametric equation of a 2D curve having axial symmetry, by applying a suitable matrix transformation we can get the parametric equation of the 3D curve that results of the "rotation" of the base 2D curve, as imbedded in 3D space, by the angle ϕ around the axis of symmetry. In case of the cardioid like curves generated with three circles in Section 4, for the curve $g_{r_2,r} : [0, 2\pi] \rightarrow \mathbb{R}^3$ defined by $g_{r_2,r}(t) = (g_1(t), g_2(t), 0)$, a simple calculation shows that the 3D curve obtained when this curve is rotated by the angle ϕ around the x–axis is

$$g_{r_2,r,\phi}(t) = \begin{bmatrix} g_1(t) \\ g_2(t)\cos(\phi) \\ g_2(t)\sin(\phi) \end{bmatrix}, \quad t \in [0, 2\pi]$$

The corresponding surface of revolution is obtained by considering variable t and parameter ϕ as two new valuables, say u and v, respectively:

$$S_{r_2,r}(u, v) = \begin{bmatrix} g_1(u) \\ g_2(u)\cos(v) \\ g_2(u)\sin(v) \end{bmatrix}, \quad u \in [0, 2\pi],\ v \in [0, \pi]$$

In Figure 13 we graph the 3D curve, and the respective surface of revolution, corresponding to the curve generated by three circles when circle C_3 with center in $E(r = 5; \gamma = 0)$, turns exteriorly around the circle C_2 with radius $r_2 = 3$. Exploration can be found in [S10].

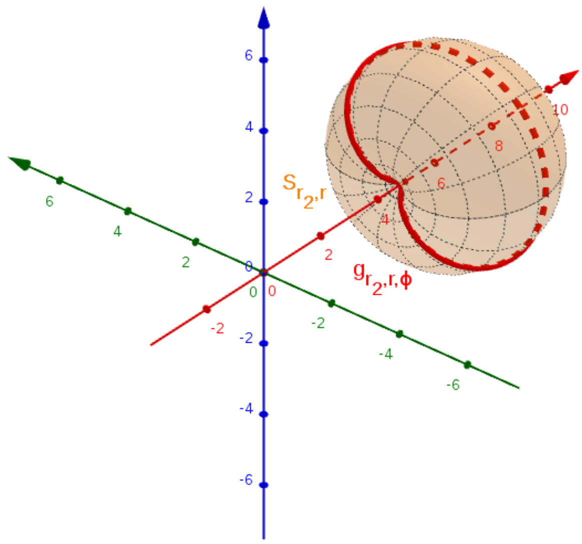

Figure 13. Heart shaped surface obtained by rotation.

5.2 Surface generated by projection

To simplify the subject, we consider in this section only ellipsoids of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a \geq b \geq c$. For the case $a > b = c$, the intersection of the corresponding ellipsoid of revolution with the plane $z = 0$ is the ellipse $x^2/a^2 + y^2/b^2 = 1$, that can be obtained as the locus of a point F moving along a circle C_2 centered at a moving point B and with radius $r_2 = (a - b)/2$, which rotates exteriorly tangent to the circle C_1 centered at the origin and with radius $r_1 = b$, as shown in Example 2 of Section 2.1. Of course, this is also valid for the projection of this ellipsoid on the plane $y = 0$. So, we can consider the revolution ellipsoid as "generated" by the sphere S_2 centered at a moving point B and radius $r_2 = (a - b)/2$, which rotates tangent exteriorly to the sphere S_1 centered at the origin and with radius $r_1 = b$. Explicitly, extending the construction in Section 2.1, as first step we let the sphere S_2 rotates in "counterclockwise" direction, around the sphere S_1 along its equator

$$C_1(t) = \begin{bmatrix} r_1 \cos(t) \\ r_1 \sin(t) \\ 0 \end{bmatrix}, \quad t \in [0, 2\pi]$$

while a point F rotates in "clockwise" direction on the sphere S_2 along its equator

$$C_2(t) = B(t) + \begin{bmatrix} r_2 \cos(t) \\ r_2 \sin(t) \\ 0 \end{bmatrix}, \quad t \in [0, 2\pi]$$

Then, as second step we let the sphere S_2 rotates in "counterclockwise" direction around the sphere S_1 along its meridian

$$C_1'(t) = \begin{bmatrix} r_1 \cos(t) \\ 0 \\ r_1 \sin(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

while a point F' rotates in "clockwise" direction on the sphere S_2 along its meridian

$$C_2'(t) = B'(t) + \begin{bmatrix} r_2 \cos(t) \\ 0 \\ r_2 \sin(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

As shown in Figure 14, the locus generated by F and F' are projections of the ellipsoid of revolution considered. Exploration can be found in [S11].

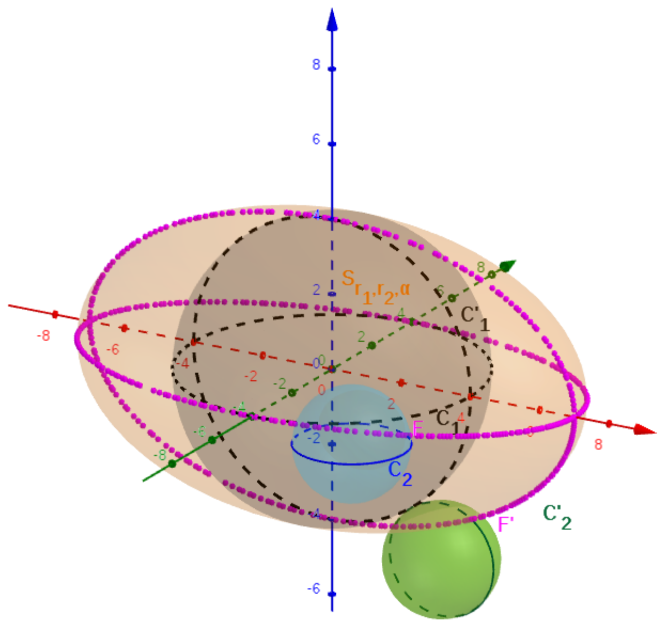

Figure 14. Projections of an ellipsoid of revolution.

A natural question is, ¿Is it possible to use two tangent spheres in order to generate the projections of an ellipsoid with axes of length $2a > 2b > 2c$?

We found that if, after applying the first step described previously, we let the sphere S_2 rotates in "counterclockwise" direction tangent interiorly to the sphere S_1 along its meridian

$$C_1''(t) = \begin{bmatrix} 0 \\ r_1\cos(t) \\ r_1\sin(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

while a point F'' rotates in "clockwise" direction on the sphere S_2 along its meridian

$$C_2''(t) = B''(t) + \begin{bmatrix} 0 \\ r_2\cos(t) \\ r_2\sin(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

then, as shown in Figure 15, the locus generated by F and F' are the projections of the ellipsoid

$$\frac{x^2}{(r_1 + 2r_2)^2} + \frac{y^2}{r_1^2} + \frac{z^2}{(r_1 - 2r_2)^2} = 1$$

(exploration can be found in [S12]).

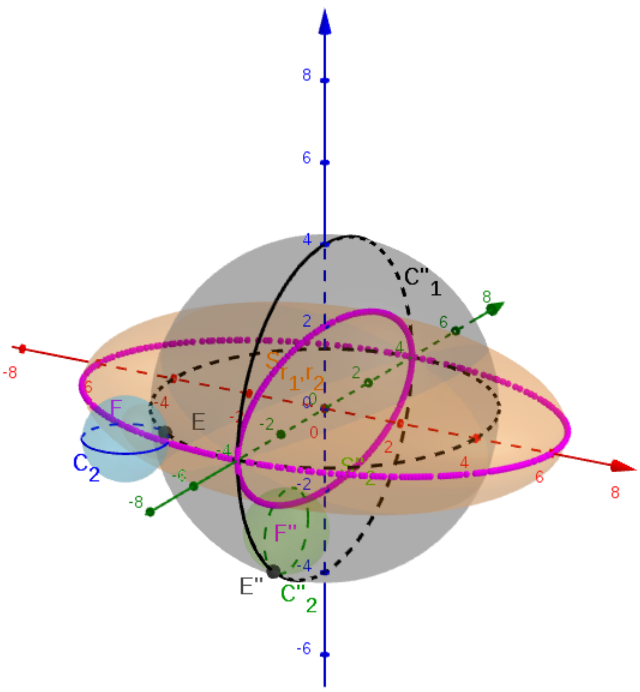

Figure 15. Projections of an ellipsoid not of revolution.

6 Future Explorations

In view of the locus problems we discussed involving circles, we would like to explore if we can replace a fixed circle by an ellipse. First, we recall the following applications, which arise from optics in Physics area. In the differential geometry of curves, the evolute of a curve is the locus of all its centers of curvature. Equivalently, it is the envelope of the normals to a curve. Caustic is the evolute of the orthotomic curve. Caustic is also the locus of all its centers of curvature of orthotomic curve or the envelope of the orthotomic normals. For example, if we are given an ellipse in blue in Figure 16. The diamond shape-like curve in red is called the caustic for the given ellipse. In other words, it is the locus of all its centers of curvature shown in D'.

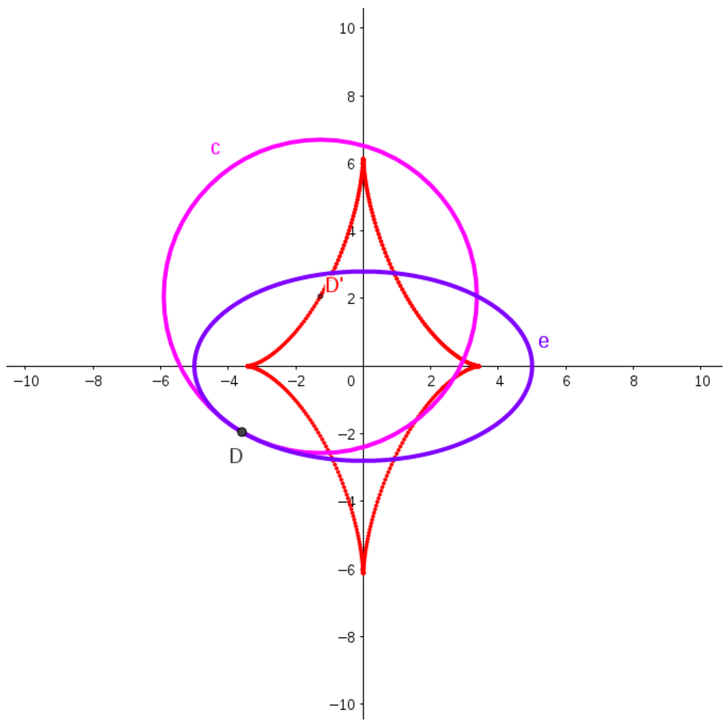

Figure 16. Caustic on an ellipse.

We shall consider the locus curve(s) resulted from the following steps:

1. We are given a fixed ellipse shown in blue in above figure.
2. We pick a moving point D on the blue ellipse, which we know that the purple circle shows the circle of curvature at various point D.
3. At each circle of curvature, we pick another point F that is different from D.
4. If D travels in clockwise direction, we would like to find the locus F that is traveling in counterclockwise direction.

5. If D travels in counterclockwise direction, we would like to find the locus F that is traveling in clockwise direction.

Just as we have discussed in Section 5, which we extended a 2D case using two circles to a 3D ellipsoid. Another area we will explore is to generate a 3D surface by using three circles as we have discussed in Section 4.

7 Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as ClassPad Manager [2] and Geometry Expressions [4] in Sections 2 and 3, or GeoGebra [3] in Sections 4 and 5. In the meantime, we use a CAS such as Maple [7] in Sections 2 and 3 or maxima [8] in Sections 4 and 5, for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school. Others require more advanced mathematics such as university levels, which are excellent examples for professional trainings for future teachers.

Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

8 Acknowledgements

The author would like to thank Qiuxia Li for providing the proofs of some examples and numerous inspiring discussions about the level of high school math content knowledge from her teaching experiences in China.

9 Supplementary Electronic Materials

[S1] `s1-Section1.ggb` (GeoGebra worksheet for Section 1).

[S2] `s2-Section2.mw` (Maple worksheet for Section 2).

[S3] `s3-Example4.mw` (Maple worksheet for Example 4).

[S4] `s4-Section4.1.ggb` (GeoGebra worksheet for Section 4.1).

[S5] `s5-Section4.1.wxm` (wxMaxima worksheet for Section 4.1).

[S6] `s6-Section4.2.ggb` (GeoGebra worksheet for Section 4.2).

[S7] `s7-Section4.2.wxm` (wxMaxima worksheet for Section 4.2).

[S8] `s8-Section5.1a.ggb` (GeoGebra worksheet for Section 5.1 ellipsoid).

[S9] `s9-Section5.1a.wxm` (wxMaxima worksheet for Section 5.1 ellipsoid).

[S10] `s10-Section5.1b.ggb` (GeoGebra worksheet for Section 5.1 cardioid like curve).

[S11] `s11-Section5.2a.ggb` (GeoGebra worksheet for Section 5.2 ellipsoid of revolution).

[S12] `s12-Section5.2b.ggb` (GeoGebra worksheet for Section 5.2 ellipsoid not of revolution).

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